

The dual model for an Ising model with nearest and next-nearest neighbors

Abstract. We construct and analyse a dual model to the Ising model with the nearest and next-nearest neighbors on the rectangular lattice (NNNI model). The Hamiltonian of the dual model turns out to contain two- and four-spin interactions. The free fermion approximation suggests that an increase in the critical temperature of the dual model caused by the four-spin interactions is limited to a finite range.

Keywords: Ising model, duality, selfduality, four-spin interaction

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1. Introduction

Duality [1, 2, 3, 4] is an important symmetry of Ising models, as it implies a relationship between the partition functions of two Ising models with Hamiltonians $\mathcal{H}_1(J)$ and $\mathcal{H}_2(K)$

$$\frac{Z_1(\mathcal{H}_1(J))}{f_1(J)} = \frac{Z_2(\mathcal{H}_2(K))}{f_2(K)}, \quad (1.1)$$

where f_1 and f_2 are some nonsingular functions of interaction constants J, K appearing in Hamiltonians \mathcal{H}_1 and \mathcal{H}_2 , respectively. Similar functional relations between the partition functions of two different Ising models can be also obtained with the help of the decoration, star-triangle or star-square transformations [4]. Duality combined with those transformations enables one to derive the expressions for the critical temperature for a wide range of Ising models. It should be stressed, however, that in the case of duality the total number of spin interactions must be the same for both Hamiltonians, \mathcal{H}_1 and \mathcal{H}_2 , which allows one to introduce the concept of dual lattice. For example, the triangular lattice turns out dual to the honeycomb lattice and the rectangular lattice is dual to itself. Using the concept of dual lattice, the duality relation can be interpreted as a relationship between high-temperature (low-temperature) expansion for the model with Hamiltonian \mathcal{H}_1 and the low-temperature (high-temperature) expansion for the model described by \mathcal{H}_2 .

The concept of duality has been extended by Wegner [5] for a wider class of Ising models M_{dn} on d -dimensional lattices characterized by a number $n = 1, 2, \dots, d$, where $n = 1$ corresponds to the Ising model with a two-spin interaction. In this paper we construct and investigate analytically the dual model to the Ising model on a rectangular lattice with the nearest- and next-nearest-neighbor interactions (NNNI model) [6, 7, 8, 9]. Although this simple extension of the Onsager model has been widely investigated for decades now, no analytical expression for the partition function of the model has been found, nor its dual model has been presented. To this end we use the usual algebraic approach and express the partition function as the trace of a power of the transfer matrix. In Section 2 we employ the classic 2D Ising model on a rectangular lattice to introduce the formalism and some basic formulas necessary to investigate the duality of a more complex model in Section 3. In particular, we show how the self-dual symmetry of this model can be related to the similarity transformation (U) of the transfer matrix. We show that the difference between two forms of the transfer matrix can be related to a change of the direction in which the matrix was constructed and present the explicit form of U . In Section 3 we apply the similarity transformation to the transfer matrix of the NNNI model. We show that there exists a simple Ising model for which the transformed matrix is the transfer matrix. The Hamiltonian of this model contains a two-spin nearest-neighbor interaction between nodes of the Brick-Wall lattice (which is topologically equivalent to a honeycomb lattice) and additional four-spin interactions (henceforth this model will be called the $(2 + 4)$ BWI model). The number of the four-spin interactions is equal to the number of bricks in the lattice. Since the two Hamiltonians have the same number of all interactions, the model can be regarded as the dual model to the NNNI model. As a result we obtain the duality relation between the partition

functions of these two models. Moreover, we show that the dual model becomes self-dual in the presence of an external magnetic field. In Section 4 the critical temperature of the $(2 + 4)$ BWI model is investigated. We show that an increase in the critical temperature of the dual model caused by the four-spin interactions is limited to a finite range. We derive the equations for both the lower and upper limits of this temperature range and find that they depend on the magnitude of the two-spin interactions only.

2. Selfduality of the Ising model on the rectangular lattice

Consider the classical Ising model with the nearest-neighbor interaction constants J_1 and J_2 (for simplicity, we assume that $J = J_1 = J_2$) on a rectangular lattice with N columns and $2M$ rows. We assume the cyclic boundary conditions in the direction of the transfer matrix action and in the perpendicular direction we adjust the boundary conditions to the transformations performed on the transfer matrix to get the result in a closed form (we adopt this convention throughout this paper). The partition function for this model reads [10, 3, 11]

$$\begin{aligned} Z_{N,2M}(\kappa) &= \text{Tr } V^N \\ &= (2 \sinh 2\kappa)^{MN} \text{Tr} \left(e^{\kappa A} e^{\tilde{\kappa} C} \right)^N, \end{aligned} \quad (2.1)$$

where $\kappa = \beta J = J/k_B T$, $\tilde{\kappa} = -\frac{1}{2} \ln \tanh \kappa$, and A and C are $2^{2M} \times 2^{2M}$ matrices defined by

$$A = \sigma_1^x + \sum_{j=1}^{2M-1} \sigma_j^x \sigma_{j+1}^x, \quad (2.2)$$

$$C = \sum_{j=1}^{2M} \sigma_j^z, \quad (2.3)$$

with

$$\sigma_j^\alpha = \underbrace{\mathbb{1} \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}}_{j-1} \otimes \sigma^\alpha \otimes \underbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}_{2M-j}, \quad (2.4)$$

where the symbol \otimes denotes the tensor product, $\mathbb{1}$ is the 2×2 unit matrix, and σ^α ($\alpha = x, y, z$) represent the Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.5)$$

and j runs through the $2M$ nodes of a lattice column.

It turns out that A and C are related to each other by a similarity transformation U ,

$$UAU^{-1} = C, \quad UCU^{-1} = A. \quad (2.6)$$

Relations of this type were found to be essential in proving the duality or self-duality of many Ising models, including the self-duality of the 1D Ising model with an external transverse magnetic field [12, 13, 14].

Applying Equation (2.6) under the trace operator in Equation (2.1), one obtains

$$\begin{aligned}
Z_{N,2M}(\kappa) &= (2 \sinh 2\kappa)^{MN} \text{Tr} \left(U \left(e^{\kappa A} e^{\tilde{\kappa} C} \right)^N U^{-1} \right) \\
&= (2 \sinh 2\kappa)^{MN} \text{Tr} \left(e^{\kappa U A U^{-1}} e^{\tilde{\kappa} U C U^{-1}} \right)^N \\
&= (2 \sinh 2\kappa)^{MN} \text{Tr} \left(e^{\kappa C} e^{\tilde{\kappa} A} \right)^N.
\end{aligned} \tag{2.7}$$

Using the cyclicity of the trace operator, Equation (2.7) can be rewritten in a well-known form

$$\begin{aligned}
Z_{N,2M}(\kappa) &= (2 \sinh 2\kappa)^{MN} \text{Tr} \left(e^{\tilde{\kappa} A} e^{\kappa C} \right)^N \\
&= \left(\frac{2 \sinh 2\kappa}{2 \sinh 2\tilde{\kappa}} \right)^{MN} (2 \sinh 2\tilde{\kappa})^{MN} \text{Tr} \left(e^{\tilde{\kappa} A} e^{\kappa C} \right)^N \\
&= \left(\frac{2 \sinh 2\kappa}{2 \sinh 2\tilde{\kappa}} \right)^{MN} Z_{N,2M}(\tilde{\kappa}) \\
&= (\sinh 2\kappa)^{2MN} Z_{N,2M}(\tilde{\kappa}),
\end{aligned} \tag{2.8}$$

which proves the self-duality of the discussed model.

Equation (2.8) was usually derived by constructing the transfer matrix along different directions of the lattice, without deriving the explicit form of U , which however will be necessary in our study further below. To obtain it, one can assume that it can be expressed as a product of two $2^{2M} \times 2^{2M}$ matrices

$$U = SR, \tag{2.9}$$

where R is a permutation matrix, whereas S exchanges Pauli matrices σ_i^x with σ_i^z and changes the signs of the interaction coefficients appropriately. A particularly simple form of R reads

$$R = \prod_{j=1}^{2M-1} R_{j,j+1}, \tag{2.10}$$

where

$$R_{j,j+1} = \underbrace{\mathbb{1} \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}}_{j-1} \otimes r \otimes \underbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}_{2M-j-1} \tag{2.11}$$

and

$$r = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

It follows that $r^{-1} = r$, $[R_{j,j+1}, R_{j+1,j+2}] \neq 0$, and

$$R^{-1} = \prod_{l=1}^{2M-1} R_{2M-l, 2M-(l-1)}. \tag{2.13}$$

The following relations

$$\begin{aligned}
R_{l,l+1}\sigma_l^x\sigma_{l+1}^xR_{l,l+1}^{-1} &= \sigma_{l+1}^x \\
R_{l,l+1}\sigma_l^xR_{l,l+1}^{-1} &= \sigma_l^x \\
R_{l,l+1}\sigma_{l+1}^xR_{l,l+1}^{-1} &= \sigma_l^x\sigma_{l+1}^x \\
R_{l,l+1}\sigma_l^z\sigma_{l+1}^zR_{l,l+1}^{-1} &= -\sigma_l^z \\
R_{l,l+1}\sigma_l^zR_{l,l+1}^{-1} &= -\sigma_l^z\sigma_{l+1}^z \\
R_{l,l+1}\sigma_{l+1}^zR_{l,l+1}^{-1} &= \sigma_{l+1}^z
\end{aligned} \tag{2.14}$$

can be used to show how A and C transform under R ,

$$\begin{aligned}
RAR^{-1} &= \sum_{j=2}^{2M} \sigma_j^x, \\
RCR^{-1} &= \sum_{j=1}^{2M-1} \sigma_l^z\sigma_{j+1}^z + \sigma_{2M}^z.
\end{aligned} \tag{2.15}$$

The cyclic boundary conditions in A are mathematically troublesome. However, since the thermodynamics of the system does not depend on the boundary conditions in the thermodynamic limit, they can be neglected [12, 13]. As for S , it can be expressed as

$$S = \mathcal{P} \prod_{j=1}^{2M} s_j, \tag{2.16}$$

where

$$s_j = \underbrace{\mathbb{1} \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}}_{j-1} \otimes s \otimes \underbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}_{2M-j}, \tag{2.17}$$

with

$$s = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{2.18}$$

whereas \mathcal{P} is a transformation exchanging matrices σ_i^α with σ_{2M-i+1}^α ,

$$\mathcal{P} = \prod_{i=1}^M P_{i,2M-i+1}, \tag{2.19}$$

where

$$P_{i,j} = \frac{1}{2} (1 + \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z) \tag{2.20}$$

and

$$P_{i,j} \sigma_i^\alpha P_{i,j}^{-1} = \sigma_j^\alpha, \quad \alpha = x, y, z \text{ and } i, j = 1, 2, \dots, 2M. \tag{2.21}$$

On applying the full transformation $U = SR$ to A and C , one arrives at Equation (2.6).

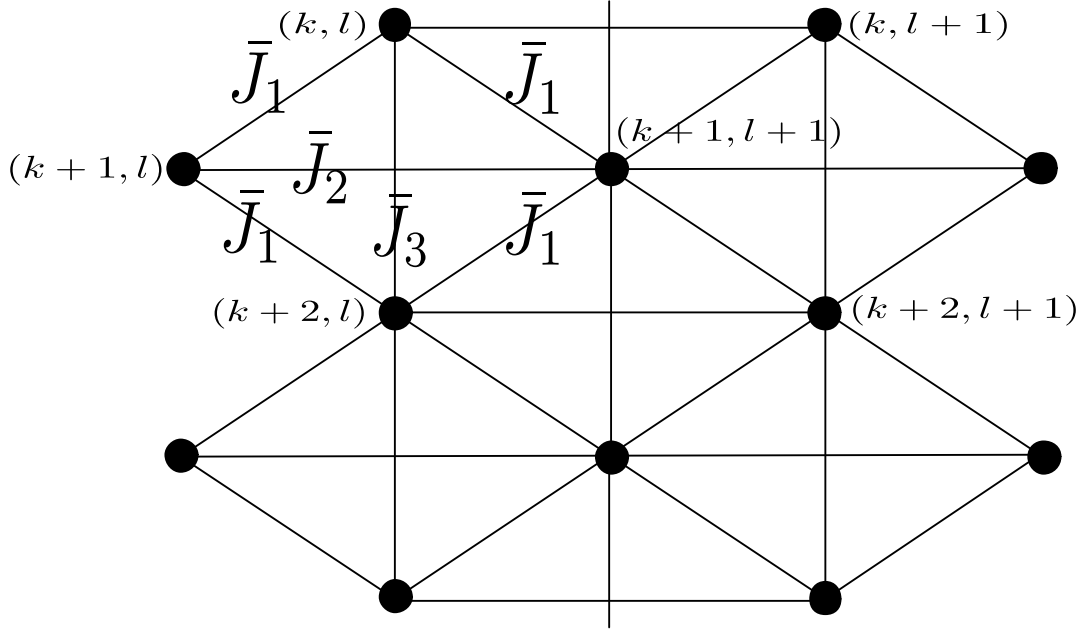


Figure 1. The Ising model with the nearest and next-nearest neighbor interactions on a rhomboidal lattice (the NNNI model). \bar{J}_1 , \bar{J}_2 , and \bar{J}_3 are the interaction constants for the diagonal, horizontal and vertical directions, respectively.

3. The dual model to the NNNI Model

The transfer matrix for the Ising model with the nearest and next-nearest neighbor interactions was found in 1956 by Temperley [15], who constructed it along the diagonal of the basic rectangular lattice defined by the nearest-neighbor interactions. Although this matrix is not of the lowest possible rank, its form is particularly simple. We will use this matrix for the rhomboidal lattice presented in Figure 1. While in most papers devoted to the NNNI model only two interaction constant are used [7, 8], $\bar{J}_1, \bar{J}_2 = \bar{J}_3$, here we consider the general case with three arbitrary constants controlling the ferromagnetic interactions $\bar{J}_1, \bar{J}_2, \bar{J}_3$ ($\bar{J}_i < 0$).

The transfer matrix and the partition function for this model on the rhomboidal lattice consisting of $2M \times N$ nodes satisfy

$$\frac{Z_{N,2M}}{(2 \sinh 2L_2)^{NM}} = \text{Tr} \left(e^{L_1 A} e^{L_3 B_{2Z-1}} e^{\tilde{L}_2 C_{2Z}} e^{L_1 A} e^{L_3 B_{2Z}} e^{\tilde{L}_2 C_{2Z-1}} \right)^N, \quad (3.1)$$

where $L_i = \beta \bar{J}_i$ ($i = 1, 2, 3$), $\tilde{L}_2 = -\frac{1}{2} \ln \tanh L_2$, A , B_{2Z} , B_{2Z-1} , C_{2Z} , and C_{2Z-1} are

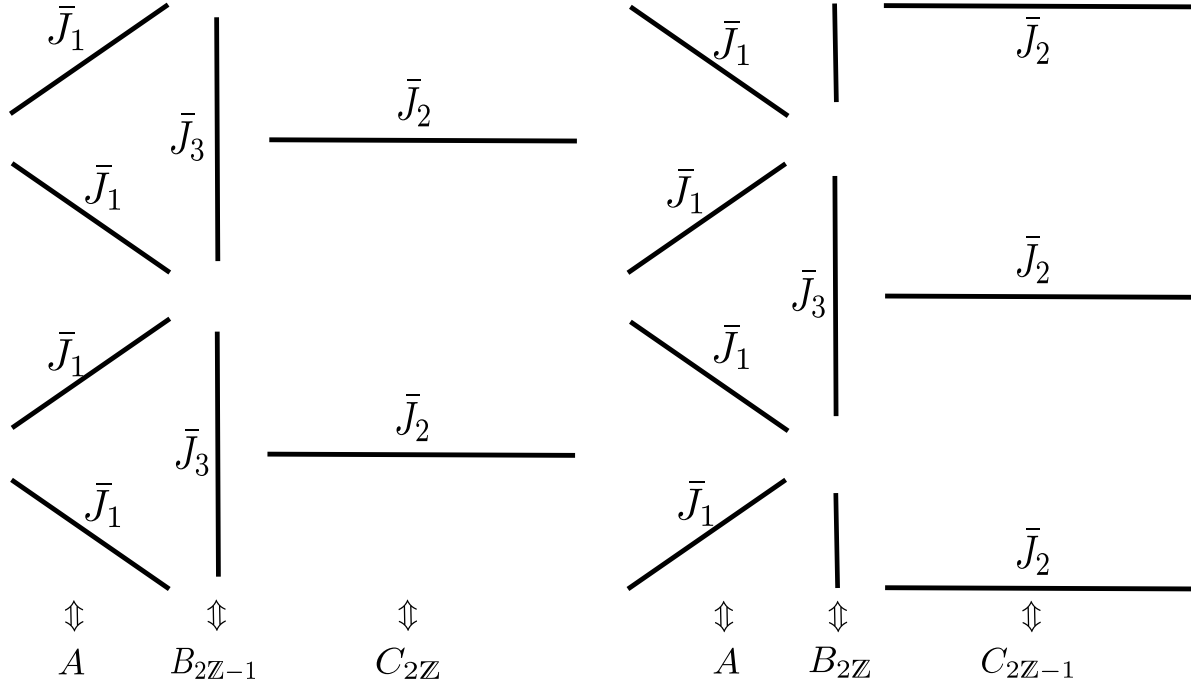


Figure 2. Correspondence between the two-spin interactions and the matrix operators.

matrices of size $2^{2M} \times 2^{2M}$, defined as follows,

$$A = \sigma_1^x + \sum_{k=1}^{2M-1} \sigma_k^x \sigma_{k+1}^x, \quad (3.2)$$

$$B_{2\mathbb{Z}} = \sigma_2^x + \sum_{k=1}^{M-1} \sigma_{2k}^x \sigma_{2k+2}^x, \quad (3.3)$$

$$B_{2\mathbb{Z}-1} = \sum_{k=1}^{M-1} \sigma_{2k-1}^x \sigma_{2k+1}^x + \sigma_1^x \sigma_{2M-1}^x \sigma_{2M}^x, \quad (3.4)$$

$$C_{2\mathbb{Z}} = \sum_{k=1}^M \sigma_{2k}^z, \quad (3.5)$$

$$C_{2\mathbb{Z}-1} = \sum_{k=1}^M \sigma_{2k-1}^z, \quad (3.6)$$

with $2\mathbb{Z}$ denoting the set of even numbers. The correspondence between these matrices and the interaction constants are shown in Figure 2. Similar transfer matrices for the NNNI model were already used in [16], however, without giving their explicit forms.

On applying the similarity transformation U to the matrices of the NNNI model we arrive

at

$$UAU^{-1} = \sum_{k=1}^{2M} \sigma_k^z = C, \quad (3.7)$$

$$UB_{2\mathbb{Z}-1}U^{-1} = \sum_{k=2}^M \sigma_{2k-2}^z \sigma_{2k-1}^z + \sigma_1^z \sigma_{2M}^z = D_{2\mathbb{Z}}, \quad (3.8)$$

$$UC_{2\mathbb{Z}}U^{-1} = \sigma_1^x + \sum_{k=1}^{M-1} \sigma_{2k}^x \sigma_{2k+1}^x = A_{2\mathbb{Z}}, \quad (3.9)$$

$$UB_{2\mathbb{Z}}U^{-1} = \sum_{k=1}^M \sigma_{2k-1}^z \sigma_{2k}^z = D_{2\mathbb{Z}-1}, \quad (3.10)$$

$$UC_{2\mathbb{Z}-1}U^{-1} = \sum_{k=1}^M \sigma_{2k-1}^x \sigma_{2k}^x = A_{2\mathbb{Z}-1}, \quad (3.11)$$

which also defines $A_{2\mathbb{Z}}$, $A_{2\mathbb{Z}-1}$, $D_{2\mathbb{Z}}$, and $D_{2\mathbb{Z}-1}$. These relations lead to

$$\frac{Z_{N,2M}}{(2 \sinh 2L_2)^{NM}} = \text{Tr } \tilde{V}^N, \quad (3.12)$$

where

$$\tilde{V} = e^{L_1 C} e^{L_3 D_{2\mathbb{Z}}} e^{\tilde{L}_2 A_{2\mathbb{Z}}} e^{L_1 C} e^{L_3 D_{2\mathbb{Z}-1}} e^{\tilde{L}_2 A_{2\mathbb{Z}-1}}. \quad (3.13)$$

Equation (3.12) ensures that if there exists an Ising model whose transfer matrix is equal to \tilde{V} , this model will be dual to the NNNI model. This is a nontrivial requirement, as while the transfer matrix of any Ising model can be expressed as a product of exponential matrices, the product of exponential matrices is hardly ever the transfer matrix of an Ising model. In the case considered here, however, such a model exists and is described by the Hamiltonian

$$\mathcal{H} = - \sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j - J_4 \sum_{\langle i,j,k,l \rangle} \sigma_i \sigma_j \sigma_k \sigma_l \quad (3.14)$$

defined on the Brick-Wall lattice [17], which is topologically equivalent to the honeycomb lattice. As explained in Figure 3, J_4 is the four-spin interaction constant for the four spins occupying the right-hand side of a brick, and the values of the nearest-neighbor interaction constants J_{ij} are non-zero only if i and j are on the same side of a brick, in which case they are equal J_1 and J_2 for interactions along the vertical and horizontal direction, respectively. All interaction constants are ferromagnetic in nature ($J_1, J_2, J_4 < 0$).

The dual lattice is composed of $2M$ horizontal chains, each containing $2N$ nodes. The partition function of the model satisfies

$$\begin{aligned} & \frac{Z_{2N,2M}(K_2, K_4, K_1)}{2^{2MN} [\sinh^2 2K_2 (e^{4K_4} \cosh^2 2K_2 - 1)]^{MN/2}} = \\ & = \text{Tr} \left(e^{\tilde{K}_2 C} e^{\tilde{K}_4 D_{2\mathbb{Z}}} e^{K_1 A_{2\mathbb{Z}}} e^{\tilde{K}_2 C} e^{\tilde{K}_4 D_{2\mathbb{Z}-1}} e^{K_1 A_{2\mathbb{Z}-1}} \right)^N, \end{aligned} \quad (3.15)$$

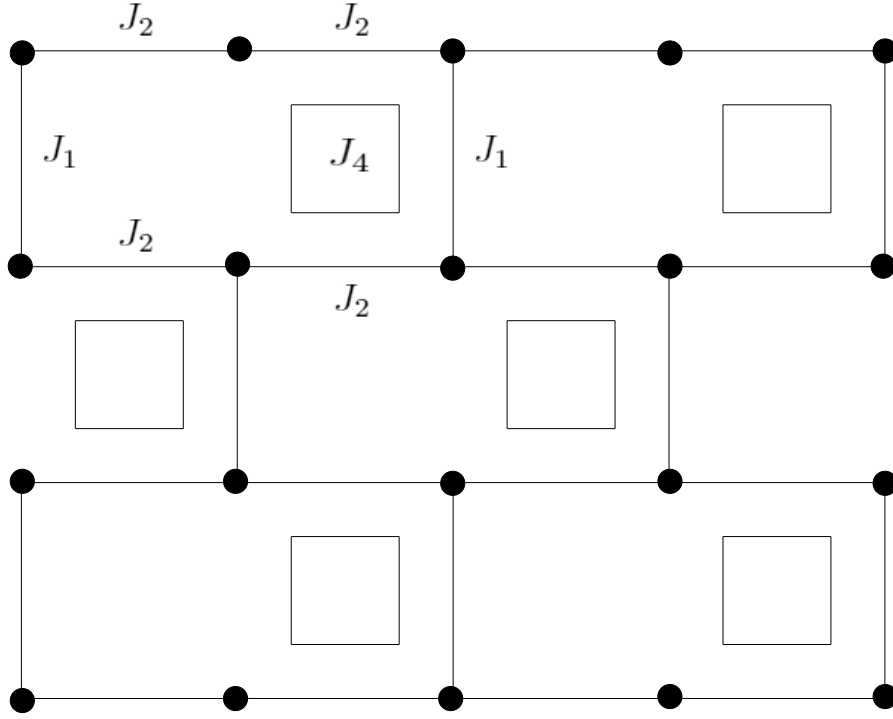


Figure 3. The (2 + 4) BWI Model.

where $K_i = \beta J_i$, ($i = 1, 2, 4$) and

$$\tilde{K}_2 = -\frac{1}{4} \ln \frac{\cosh 2K_2 - e^{-2K_4}}{\cosh 2K_2 + e^{-2K_4}}, \quad (3.16)$$

$$\tilde{K}_4 = -\frac{1}{4} \ln \frac{\cosh^2 2K_2 - 1}{\cosh^2 2K_2 - e^{-4K_4}}. \quad (3.17)$$

The transfer matrix in Equation (3.15) acts on the vertical groups of nodes and advances the partition function in the horizontal direction [18]. Its explicit form can be obtained from Onsager's classical transfer matrix for the rectangular lattice [3], see Equation (2.1). First, we add the four-spin interactions that modify only two horizontal two-spin bonds lying one above the other. Next, half of the vertical two-spin interactions connecting the horizontal chains has to be removed. While adding the four-spin interactions to the transfer matrix, we can use a relationship

$$\Delta^{1/4} e^{\tilde{K}_2(\sigma_k^z + \sigma_{k+1}^z)} e^{\tilde{K}_4 \sigma_k^z \sigma_{k+1}^z} = \cosh K_4 c_k c_{k+1} + \sinh K_4 \sigma_k^x \sigma_{k+1}^x c_k c_{k+1} \sigma_k^x \sigma_{k+1}^x, \quad (3.18)$$

where $\Delta = \frac{1}{16} \sinh^2 2K_2 (e^{4K_4} \cosh^2 2K_2 - 1)$ and

$$c_k = \underbrace{\mathbb{1} \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}}_{k-1} \otimes c \otimes \underbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}_{2M-k}, \quad (3.19)$$

$$c = \begin{pmatrix} \cosh K_2 & 0 \\ 0 & \sinh K_2 \end{pmatrix}, \quad (3.20)$$

and $k = 1, \dots, 2M$. Finally, we find the duality relationship between the NNNI model, Figure 1, and the (2 + 4) BWI model, Figure 3,

$$\frac{Z_{N,2M}(L_1, L_3, L_2)}{(2 \sinh 2L_2)^{MN}} = \frac{Z_{2N,2M}(K_2, K_4, K_1)}{2^{2MN} [\sinh^2 2K_2 (e^{4K_4} \cosh^2 2K_2 - 1)]^{MN/2}}, \quad (3.21)$$

where the relations between the interaction constants for both model are given by

$$L_1 = \tilde{K}_2, \quad L_3 = \tilde{K}_4, \quad \tilde{L}_2 = K_1. \quad (3.22)$$

4. The critical temperature for the (2 + 4) BWI model

While the Ising model on the Brick-Wall lattice can be solved exactly [19], addition of the four-spin interaction makes it intractable analytically. However, it turns out that it is still possible to draw some important conclusions about the way the critical temperature of the Ising model defined by Hamiltonian (3.14) depends on the strength of the four-spin interactions.

In the Ising model with nearest and next-nearest neighbor interactions (NNNI model, Figure 1) there are two important limiting cases: $\bar{J}_3 = 0$ and $\bar{J}_1 = 0$. In the former case the NNNI model reduces to the Ising model with nearest-neighbor interactions on the triangular lattice with the interaction constants (\bar{J}_1, \bar{J}_2) . Since in this case $L_3 = \beta \bar{J}_3 = 0$, matrices $e^{L_3 D_{2Z}}$ and $e^{L_3 D_{2Z}-1}$ appearing in Equation (3.13) become the identity matrices and the critical point of this model can be determined exactly. It is worth noticing that the duality transformation converts the transfer matrix of this model into the transfer matrix of the Ising model with two nearest-neighbor interactions on a honey comb lattice equivalent to the Brick-Wall lattice. This explains why the dual model to NNNI model is defined on the Brick-Wall lattice. In the second case ($\bar{J}_1 = 0$), the NNNI model splits into two identical independent self-dual Ising models with nearest-neighbor interactions on the rectangular lattice. Similarly to the previous case, matrices $e^{L_1 A}$ in Equation (3.13) become the identity matrices and the critical point for this model is exactly determinable.

Equation (3.22), which describes relationships between the interactions of the NNNI and (2+4) BWI models, reduces to

$$0 = L_3 = \tilde{K}_4 = -\frac{1}{4} \ln \frac{\cosh^2 2K_2 - 1}{\cosh^2 2K_2 - e^{-4K_4}} \quad (4.1)$$

for $\bar{J}_3 = 0$ and to

$$0 = L_1 = \tilde{K}_2 = -\frac{1}{4} \ln \frac{\cosh 2K_2 - e^{-2K_4}}{\cosh 2K_2 + e^{-2K_4}} \quad (4.2)$$

for $\bar{J}_1 = 0$. Equation (4.1) is equivalent to $K_4 = 0$, whereas Equation (4.2) can be satisfied only in the limit of $K_4 \rightarrow \infty$. The critical temperatures satisfy [4]

$$e^{-4 \frac{J_4}{kT_C}} = \cosh^2 \left(\frac{2J_2}{kT_C} \right) \frac{\tanh^2 \left(\frac{J_1}{kT_C} \right)}{\tanh^2 \left(\frac{J_1}{kT_C} \right) + 1}, \quad (4.3)$$

$$e^{-4 \frac{J_4}{kT_C}} = \cosh^2 \left(\frac{2J_2}{kT_C} \right) - e^{4 \frac{J_1}{kT_C}} \sinh^2 \left(\frac{2J_2}{kT_C} \right), \quad (4.4)$$

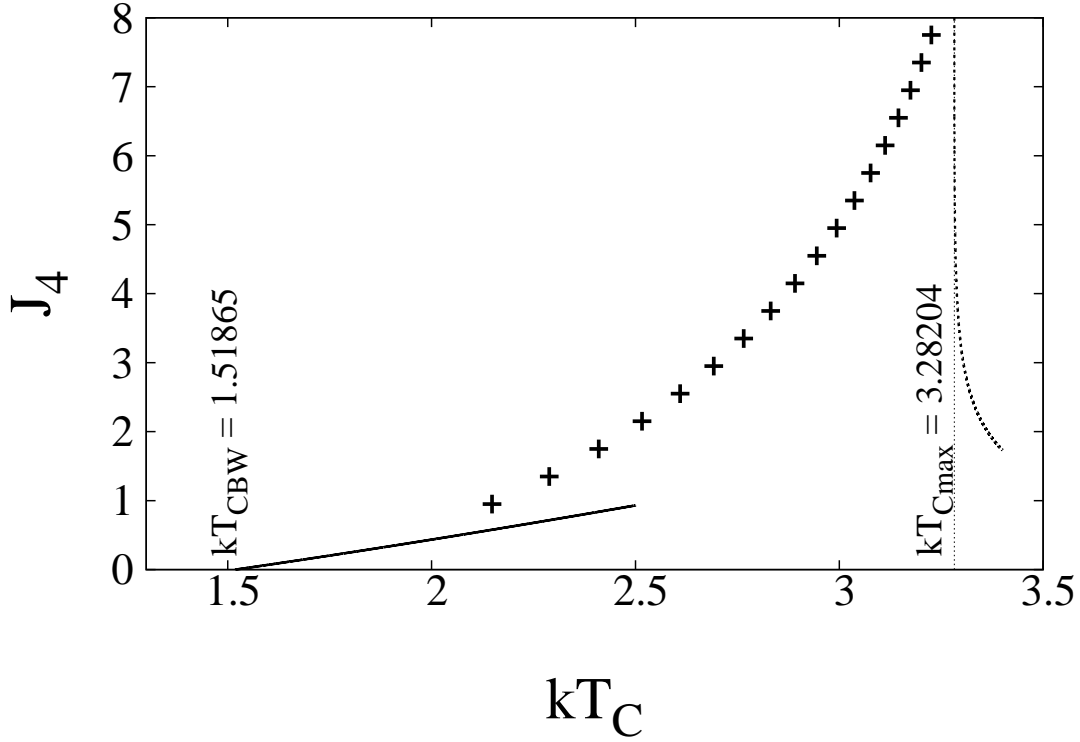


Figure 4. Relation between the critical temperature (kT_C) and the four-spin interaction constant (J_4) in the dual model with $J_1 = J_2 = 1$. The solid and dotted lines are the solution of Eqs. (4.3) and (4.4), respectively, and the crosses indicate a possible location of the phase transition. kT_{CBW} is the critical temperature for the Ising model on a Brick-Wall lattice and kT_{Cmax} is the maximum critical temperature for the ferromagnetic case of the $(2 + 4)$ BWI model.

for $\bar{J}_3 = 0$ and $\bar{J}_1 = 0$, respectively.

Figure 4 presents a sketch of the relation between J_4 and the critical temperature kT_C in a particular case of $J_1 = J_2 = 1$. The solid line represents the asymptotic solution for the case $K_4 \rightarrow 0$ obtained from Equation (4.3), and the dotted line depicts the asymptotic solution of Equation (4.4) obtained for $K_4 \rightarrow \infty$. The solution of Equation (4.4) has only the left-hand vertical asymptote at the temperature kT_{Cmax} given as the solution to

$$\cosh\left(\frac{2}{kT_{Cmax}}\right) = \exp\left(\frac{2}{kT_{Cmax}}\right) \sinh\left(\frac{2}{kT_{Cmax}}\right), \quad (4.5)$$

which yields $kT_{Cmax} \approx 3.28204$. The fact that Equation (4.4) has only the left-hand side asymptote at the critical point defined by $K_4 \rightarrow \infty$ is in contradiction to the Griffith's inequalities [20], which imply that if an additional ferromagnetic interaction is introduced into a ferromagnetic Ising model, the critical temperature T_c will increase with the magnitude of this additional interaction. Therefore, this solution has to be rejected as nonphysical. Our preliminary computer simulations (which will be discussed elsewhere) suggest that the relation between J_4 and T_C is located along the line marked with crosses in Figure 4, with the right vertical asymptote at the point given by Equation (4.5).

Our considerations show that the additional four-spin interaction in the Ising model on a

Brick Wall lattice does not yield an infinite increase of the critical temperature. Instead, the critical temperature is always limited by a condition involving two-spin interactions only. In order to confirm this qualitative result obtained for a particular choice of J_1 and J_2 , we plan to perform detailed computer simulations of the model.

5. Self-duality of the (2 + 4) BWI model in the presence of an external magnetic field

As in other pairs of Ising models, the inclusion of an external magnetic field makes the duality relationship between the NNNI and the (2 + 4) BWI models cease to exist: duality relations (3.21) – (3.22) were obtained only in the absence of an external magnetic field. However, the Ising model described by the Hamiltonian (3.14) has an additional internal symmetry. If one adds to the Hamiltonian an interaction with an external magnetic field $-H \sum_i \sigma_i$, then the partition function will take the form

$$\frac{Z_{2N,2M}(K_2, K_4, h, K_1)}{2^{2MN} [\sinh^2 2K_2 (e^{4K_4} \cosh^2 2K_2 - 1)]^{MN/2}} = \text{Tr}(V)^N, \quad (5.1)$$

where

$$V = e^{\tilde{K}_2 C} e^{\tilde{K}_4 D_{2\mathbb{Z}}} e^{hE} e^{K_1 A_{2\mathbb{Z}}} e^{\tilde{K}_2 C} e^{\tilde{K}_4 D_{2\mathbb{Z}-1}} e^{hE} e^{K_1 A_{2\mathbb{Z}-1}} \quad (5.2)$$

is the transfer matrix, $h = \beta H$, and E is a matrix defined as

$$E = \sum_{k=1}^{2M} \sigma_k^x. \quad (5.3)$$

Let

$$\mathcal{S} = \prod_{i=1}^{2M} s_i, \quad (5.4)$$

where s_i are given by Equation (2.17). This operator exchanges σ_k^x with σ_k^z . Applying it as a similarity transformation for V leads to

$$V' = \mathcal{S} V \mathcal{S}^{-1}, \quad (5.5)$$

where

$$V' = e^{\tilde{K}_2 E} e^{\tilde{K}_4 A_{2\mathbb{Z}}} e^{hC} e^{K_1 D_{2\mathbb{Z}}} e^{\tilde{K}_2 E} e^{\tilde{K}_4 A_{2\mathbb{Z}-1}} e^{hC} e^{K_1 D_{2\mathbb{Z}-1}}. \quad (5.6)$$

Thus, V' can be interpreted as the transfer matrix of the (2 + 4) BWI model in which the interactions have been changed as follows,

$$\begin{aligned} h &\rightarrow \tilde{K}_2, \\ K_1 &\rightarrow \tilde{K}_4. \end{aligned} \quad (5.7)$$

The second difference is that while in the initial Ising model the four-spin interactions act on the right-hand side of the bricks, the four-spin interactions appear on their left-hand sides of the transformed model. The self-duality relation for this model is described by the equation

$$Z_{2N,2M}(K_2, K_4, h, K_1) = \alpha Z_{2N,2M}(\tilde{K}_2, \tilde{K}_4, \tilde{h}, \tilde{K}_1), \quad (5.8)$$

where

$$\alpha = \left[\frac{\sinh^2 2K_2 (e^{4K_4} \cosh^2 2K_2 - 1)}{\sinh^2 2\tilde{h} (e^{4\tilde{K}_1} \cosh^2 2\tilde{h} - 1)} \right]^{\frac{MN}{2}} \quad (5.9)$$

and

$$\begin{aligned} \tilde{K}_2 &= -\frac{1}{4} \ln \frac{\cosh 2K_2 - e^{-2K_4}}{\cosh 2K_2 + e^{-2K_4}}, \\ \tilde{K}_4 &= -\frac{1}{4} \ln \frac{\cosh^2 2K_2 - 1}{\cosh^2 2K_2 - e^{-4K_4}}, \\ \tilde{K}_1 &= -\frac{1}{4} \ln \frac{\cosh^2 2h - 1}{\cosh^2 2h - e^{-4K_1}}, \\ \tilde{h} &= -\frac{1}{4} \ln \frac{\cosh 2h - e^{-2K_1}}{\cosh 2h + e^{-2K_1}}. \end{aligned} \quad (5.10)$$

Unfortunately the model under consideration is a ferromagnetic one, so the phase transition in an external magnetic field does not exist. Self-duality enables one only to find a number of interesting relations between correlations function for the $(2 + 4)$ BWI model.

6. Summary

The main goal of this paper was to find and investigate the dual model for the 2D Ising model with the isotropic nearest and anisotropic next-nearest neighbor interactions on a rectangular lattice. The dual model turned out to be a 2D Ising model with two-spin nearest-neighbor anisotropic interactions and additional four-spin interactions on the Brick-Wall lattice. The appearance of the four-spin interactions is associated with the presence of non-planar next-nearest neighbor interactions in the original model. The way we constructed the dual model is quite general, does not require the introduction of the dual lattice concept and can be applied to other Ising models.

Investigation of the impact of the four-spin interactions on the critical temperature of the dual model revealed that while the four-spin interactions can increase the critical temperature of the model, this increase is restricted to a finite temperature range. Moreover, the limits of this range are bounded by the two-spin interaction constants of the dual model. This rather unexpected result has to be analysed further, e.g. by computer simulation, for example, using Landau's approach [21]. Computer simulations are also necessary to investigate the exact location of the phase transition as well as its critical properties, including the universality class. They will also be useful in relating the results for the dual model with the properties of the original Ising model.

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